3.4 The Expected Value of a Function of a Random Variable

Often times in the real world, we wish to look at a function of our variables – or transform them. Changing feet into meters, Fahrenheit into centigrade, etc. In this section we seek to determine the expected value of this new variable. If we transform a random variable  by some function , it should be clear that  itself is a random variable.

Suppose that we have a discrete random variable  with pmf given below. We now let . What is the expected value of  By definition . So to determine the , we will find  and then compute .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| x |  |  | y |  |
| -2 | 1/8 |  | 1 | 1/8 |
| -1 | 1/8 |  | 3 | 1/8 |
| 0 | 2/8 |  | 5 | 2/8 |
| 1 | 1/8 |  | 7 | 1/8 |
| 2 | 1/8 |  | 9 | 1/8 |
| 3 | 1/8 |  | 11 | 1/8 |
| 4 | 1/8 |  | 13 | 1/8 |

So, .

We also note that,



**Theorem:** *Given a discrete random variable  and the linear transformation* , .

**Proof:** *Since  is a 1 to 1 function, this will be rather straight forward. *

Note that the argument above would work with any function  that is 1 to 1.

**Theorem:** *Given a discrete random variable  and the linear transformation* , .

Proof: 



Thus, the expected value of a random variable is a **linear operator**.

**Example:** *If a random variable  has mean , determine the mean of , where* 

 or 

If we think about this last example, it seems like a no brainer that it is true. Suppose that the average score is 4. I now double everybody’s score. The new average is 8. I now add 3 to everybody’s already doubled score. The new average is 11.

In this linear case, notice if we consider , then . We would naturally wonder if that statement always holds. That is, will it always be true that If  then ? The answer is no, this is not always true. **How will we show this?**

Suppose that we have a discrete random variable  with pmf given below. The expected value of X can be determined by . We now let . What is the expected value of ? By definition . So to determine the , we will find  and then compute .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| x |  |  |  | y |  |  |  | y |  |  |
| -2 | 1/8 | -2/8 |  | 4 | 1/8 | 4/8 |  | 0 | 2/8 | 0 |
| -1 | 1/8 | -1/8 |  | 1 | 1/8 | 1/8 |  | 1 | 2/8 | 2/8 |
| 0 | 2/8 | 0 |  | 0 | 2/8 | 0 |  | 4 | 2/8 | 8/8 |
| 1 | 1/8 | 1/8 |  | 1 | 1/8 | 1/8 |  | 9 | 1/8 | 9/8 |
| 2 | 1/8 | 2/8 |  | 4 | 1/8 | 4/8 |  | 16 | 1/8 | 16/8 |
| 3 | 1/8 | 3/8 |  | 9 | 1/8 | 9/8 |  |  |  |  |
| 4 | 1/8 | 4/8 |  | 16 | 1/8 | 16/8 |  |  |  |  |

 

We note that . So, we have our counter example and in general, it is not true that . We do however see something helpful when looking at the first two groupings in the above table. We see that the probabilities are identical and that each y-value is the square of the x-value. This leads us to the following theorem that will help us determine :

**Theorem:** *Given a discrete random variable  and a function* , .

The theorem is much bigger than it seems. It allows us to determine the expected value of a transformation of a random variable without determining the distribution of the new random variable. We easily found  in the above problem. With continuous random variables, it can often be difficult to determine , so this will be huge when we get to continuous random variables.

3.5 Some Important Functions of a Random Variable (Moments and Variance)

**Definition:** *The* ***Variance*** *of a discrete random variable  is defined by the formula  (provided the sum converges). Alternatively, *.

**Definition:** *The Standard Deviation of a random variable, denoted by , is defined by* 

Note that we can consider the function . Then the variance is just .

In a data set, we use the standard deviation of the data set to denote a measure of the variation of the data. The standard deviation of a random variable does the same thing. It gives us a measure of how spread out the support is with respect to the associated probabilities.

**Example:** *Determine the standard deviation for rolling a single die.*

. So .

**Theorem:** 

**Proof:** 





**Example:** *Recalculate the variance of a single die using the theorem.*



In general, this theorem allows a much easier calculation than the definition. Soon, we will see problems where this formula is a necessity.

Not all random variables have a finite variance. We saw earlier that we can have a random variable that does not have a finite mean. We can also have a random variable with finite mean, but not finite variance.

, where k is the value that makes  a pmf. That is, so that . This random variable has finite mean since , which is finite. To determine the variance of our random variable, we need to determine .  which diverges. So, this random variable has finite mean but not finite variance.

**Theorem:** *Given a discrete random variable  and the linear transformation* , . Or, we can write 



**Exercise:** Given a random variable , with variance , determine the variance of .

**Exercise:** Given a random variable , with variance , determine the variance of .

The  is often called the **first moment** of ****and  is called the **second moment** of .

**Definition:** *The* ***kth moment*** *of  is defined to be* 

These moments are important features of a random variable. Sometimes they are called the moments about the origin and then moments about the mean would be defined as .